Purely Goldie Extending Modules

Saad A. Al-Saadi  
Ikbal A. Omer
Dep. of Mathematics /College of Science/University of Al Mustansiriyah
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Abstract

An $R$-module $M$ is extending if every submodule of $M$ is essential in a direct summand of $M$. Following Clark, an $R$-module $M$ is purely extending if every submodule of $M$ is essential in a pure submodule of $M$. It is clear purely extending is generalization of extending modules. Following Birkenmeier and Tercan, an $R$-module $M$ is Goldie extending if, for each submodule $X$ of $M$, there is a direct summand $D$ of $M$ such that $X \beta D$.

In this paper, we introduce and study class of modules which are proper generalization of both the purely extending modules and $G$-extending modules. We call an $R$-module $M$ is purely Goldie extending if, for each $X \subseteq M$, there is a pure submodule $P$ of $M$ such that $X \beta P$. Many characterizations and properties of purely Goldie extending modules are given. Also, we discuss when a direct sum of purely Goldie extending modules is purely Goldie extending and moreover we give a sufficient condition to make this property of purely Goldie extending modules is valid.

Key words: extending module, purely extending module, $G$-extending module, purely Goldie extending.
Introduction

Throughout all rings are associative and $R$ denotes a ring with identity and all modules are unitary $R$-modules. A submodule $X$ of a module $M$ is called essential if every non-zero submodule of $M$ intersects $X$ nontrivially (notionally, $X \leq^e M$). Also, a submodule $X$ of $M$ is closed in $M$, if it has no proper essential extension in $M[1]$.

Recall that a module $M$ is extending if every submodule of $M$ is essential in a direct summand of $M$. Equivalently, every closed submodule of $M$ is a direct summand [1]. Many generalizations of extending modules are extensively studied. Following Fuchs [2] and Clark [3], an $R$-module $M$ is purely extending if every submodule of $M$ is essentially a pure submodule of $M$ (recall that a submodule $N$ of an $R$-module $M$ is pure if $IM \cap N = IN$ for every finitely generated ideal $I$ of $R$). Also in [4], the following relations on the set of submodules of an $R$-module $M$ are considered. (1) $X \alpha Y$ if and only if there exists a submodule $A$ of $M$ such that $X \leq^e A$ and $Y \leq A$; (ii) $X \beta Y$ if and only if $X \cap Y \leq^e X$ and $X \cap Y \leq Y$. Following [4], $\alpha$ is reflexive and symmetric, but it may not be transitive. Also, $\beta$ is an equivalence relation. Moreover, an $R$-module $M$ is extending if and only if for each submodule $X$ of $M$, there exists a direct summand $D$ of $M$ such that $X \alpha D$. In 2009 Birkenmeier and Tercan [4], an $R$-module $M$ is called Goldie extending (shortly, $G$-extending) if, for each $X$ submodule of $M$, there is a direct summand $D$ of $M$ such that $X \beta D$.

In section one, we introduce purely $G$-extending modules. An $R$-module $M$ is $G$-extending if, for each $X \leq M$, there is a pure submodule $P$ of $M$ such that $X \beta P$. It is clear that every $G$-extending (purely extending) module is purely $G$-extending module and the converse is not true in general. Additional conditions are given to make the converse true. In fact we prove that: let $M$ be a pure split. Then $M$ is a purely $G$-extending module if and only if $M$ is a $G$-extending module. Moreover, the hereditary property of purely $G$-extending modules is discussed. We call an $R$-module $M$ is purely $G^+$-extending if every direct summand of $M$ is purely $G$-extending. We do not know whether every purely $G$-extending module is purely $G^+$-extending. Indeed, we conclude that every purely extending module is purely $G^+$-extending. Finally, we prove that an $Z$-module is extending if and only if $M$ is a purely extending and $M$ is a $G$-extending.

In section two, various characterizations of purely $G$-extending modules are given. For example, we prove that an $R$-module $M$ is purely $G$-extending if and only if every direct summand $A$ of the injective hull $E(M)$ of $M$, there exists a pure submodule $P$ of $M$ such that $(A \cap M) \beta P$. On other direction, the direct sum property of purely $G$-extending modules is discussed. We prove that, if $M_i$ is purely $G$-extending module for each $i \in I$ and every closed submodule of $M=\bigoplus_{i \in I} M_i$ is fully invariant, then $M=\bigoplus_{i \in I} M_i$ is purely $G$-extending module.

1. Purely Goldie Extending Modules.

Recall that an $R$-module $M$ is $G$-extending if, for each $X$ submodule of $M$, there is a direct summand $D$ of $M$ such that $X \beta D$. Equivalently, $M$ is Goldie extending if and only if for each closed submodule $C$ of $M$, there is a direct summand $D$ of $M$ such that $C \beta D[4]$. Also, an $R$-module $M$ is purely extending module if every submodule of $M$ is essential in a pure submodule of $M$ [3].

We introduce and study the class of modules which is a generalization of both $G$-extending modules and purely extending modules.
**Definition (1.1)**

An $R$-module $M$ is called purely Goldie extending (shortly, purely $G$-extending) if, for each $X \leq M$, there is a pure submodule $P$ of $M$ such that $X \beta P$.

**Remarks and Examples (1.2)**

1) Every purely extending module is a purely $G$-extending, but the converse is not true in general. For example, the $Z$-module $M = Z_p \oplus Q$ is a purely $G$-extending since $M$ is $G$-extending [4]. But by [4, Example (3.20)] and proposition (1.14), $M = Z_p \oplus Q$ is not purely extending $Z$-module.

2) Every $G$-extending module is purely $G$-extending, but the converse is not true in general. For example, by [5, Example (3.4)], the $Z$-module $M = \bigoplus_{i \in I} Z$ is purely extending but it is not extending. So $M$ is a purely $G$-extending while, by proposition (1.14), $M$ is not $G$-extending.

3) Every uniform module is purely $G$-extending, but the converse is not true in general. For example, $Z_6$ as $Z$-module is purely $G$-extending but it is not uniform.

Recall that an $R$-module $M$ is a pure-split if every pure submodule of $M$ is a direct summand [6]. The following proposition gives conditions under which the concepts of $G$-extending modules and purely $G$-extending modules are equivalent.

**Proposition (1.3):**

Let $M$ be a pure-split $R$-module. Then $M$ is a purely $G$-extending if and only if $M$ is a $G$-extending.

Following [7], a non-zero $R$-module $M$ is pure-simple if the only pure submodules of $M$ are 0 and $M$ itself.

**Proposition (1.4):**

Let $M$ be a pure-simple $R$-module. Then $M$ is a purely $G$-extending if and only if $M$ is a uniform module.

**Proof:** ($\Rightarrow$) Let $X$ be a submodule of $M$. By assumption, there is a pure submodule $P$ of $M$ such that $X \beta P$. So, $X \cap P$ is essential in $P$. But $M$ is a pure-simple then $P = M$, then $X$ is essential in $M$. Thus, $M$ is a uniform module.

($\Leftarrow$) Let $X$ be a submodule of $M$. Since $M$ is a uniform module, then $X$ is essential in $M$, but $M$ is a pure submodule of $M$, then $X \beta M$. Hence, $M$ is a purely $G$-extending.

**Corollary (1.5):**

Let $M$ be a pure-simple $R$-module. Then the following statements are equivalent.

1) $M$ is a purely extending module.

2) $M$ is a purely $G$-extending module.

3) $M$ is uniform module.

Following [4], a submodule of $G$-extending module need not to be $G$-extending. Moreover, a submodule of purely extending module need not to be purely extending [5]. In fact, we do not know whether a submodule of a purely $G$-extending module is purely $G$-extending. Indeed, we have the following result.

**Proposition (1.6):**

Every submodule $N$ of a purely $G$-extending $R$-module $M$ with the property that the intersection of $N$ with any pure submodule of $M$ is a pure submodule of $N$ is purely $G$-extending.
Proof: Let $A$ be a submodule of $N$. Since $M$ is a purely $G$-extending, then there is a pure submodule $P$ of $M$ such that $A \beta P$. By assumption, $P \cap N$ is a pure submodule of $N$. But, $(A \cap P) \leq P$ and $(A \cap P) \leq A$, so $(A \cap (P \cap N)) \leq (P \cap N)$ and $(A \cap (P \cap N)) \leq (A \cap N) = A$. Therefore, $A \beta (P \cap N)$. Thus, $N$ is purely $G$-extending module.

From [4], recall that $M$ is $G^+$-extending module if every direct summand of $M$ is $G$-extending. This lead us to introduce the following.

Definition (1.7):
An $R$-module $M$ is called purely $G^+$-extending if every direct summand of $M$ is purely $G$-extending.

In fact, we do not know whether, every purely $G$-extending module is purely $G^+$-extending. In fact, we have the following result.

Proposition (1.8):
Every purely extending module is purely $G^+$-extending module.

Proof: Let $N$ be a direct summand of a purely extending module $M$. By [5], $N$ is purely extending module. Hence $N$ is purely $G$-extending module. Thus, $M$ is a purely $G^+$-extending.

But the converse of proposition (1.8) is not true in general, for example, the $Z$-module $M = Z_p \oplus Q$ (for any prime number $p$) is not purely extending by (1.2), but $M$ is purely $G^+$-extending, since the only direct summands of $M$, $(Z_p \oplus 0), (0 \oplus Q)$, $(0 \oplus 0)$ and $M$, which are purely $G$-extending.

Recall that an $R$-module $M$ has the pure intersection property (PIP) if the intersection of any two pure submodule of $M$ is pure [8].

Proposition (1.9):
Let $M$ be a purely $G$-extending and $M$ has the PIP. Then $M$ is a purely $G^+$-extending.

Proof: Let $N$ be a direct summand of $M$ and $A$ be a submodule of $N$. Since $M$ is a purely $G$-extending, then there is a pure submodule $P$ of $M$ such that $A \beta P$. But $M$ satisfies PIP, then $P \cap N$ is a pure submodule of $M$. But $P \cap N \subseteq N$, hence $P \cap N$ is a pure submodule of $N$. Therefore, $A = (A \cap N) \beta (P \cap N)$ by [9], and so $M$ is a purely $G^+$-extending.

Corollary (1.10):
Let $M$ be a prime module over a Beizout domain. If $M$ is a purely $G$-extending module, then $M$ is a purely $G^+$-extending.

Recall that an $R$-module $M$ is a multiplication if for each submodule $A$ of $M$, there exists an ideal $I$ of $R$ such that $A = IM$ [10]. Since every multiplication module has the PIP [8]. Thus, we have the next corollary.

Corollary (1.11):
Let $M$ be a multiplication purely $G$-extending module. Then $M$ is a purely $G^+$-extending.

Corollary (1.12):
Let $M$ is cyclic module over a commutative ring $R$. If $M$ is a purely $G$-extending, then $N$ is purely $G$-extending.

Corollary (1.13):
Let $R$ be a purely $G$-extending commutative ring, then $R$ is a purely $G^+$-extending.
The following result gives a characterization of extending abelian groups.

Proposition (1.4):
A \( Z \)-module \( M \) is extending module if and only if \( M \) is a purely extending and \( M \) is a \( G \)-extending as \( Z \)-module.

Proof: \((\Rightarrow)\) it is clear that.

\((\Leftarrow)\) Let \( N \) be a closed submodule of \( M \). Since \( M \) is a purely extending, then \( N \) is a pure submodule of \( M \) by [5]. Also, since \( M \) is a \( G \)-extending as \( Z \)-module by [4], then \( N \) is a direct summand of \( M \). Therefore, \( M \) is extending module. ■

2. Characterizations of Purely Goldie Extending Modules

It is known that \( M \) is a purely extending module if and only if every closed submodule in \( M \) is a pure in \( M \) [5]. Also, from [4], \( M \) is \( G \)-extending module if and only if for every closed submodule \( C \) of \( M \), there is a direct summand \( D \) of \( M \) such that \( C \beta D \).

Here, we give analogous characterization of purely \( G \)-extending modules.

Proposition (2.1):
An \( R \)-module \( M \) is purely \( G \)-extending if and only if for every closed submodule \( C \) of \( M \), there is a pure submodule \( P \) of \( M \) such that \( C \beta P \).

Proof: \((\Rightarrow)\) it is clear .

\((\Leftarrow)\) Let \( A \) be a submodule of \( M \). By Zorn's lemma, there exists a closed submodule \( C \) of \( M \) such that \( A \) is essential in \( C \). So, we have \( A \beta C \). By assumption, there exists a pure submodule \( P \) of \( M \) such that \( C \beta P \). Since \( \beta \) is transitive relation, then \( A \beta P \). Therefore, \( M \) is purely \( G \)-extending module. ■

Proposition (2.2):
An \( R \)-module \( M \) is purely \( G \)-extending if and only if every direct summand \( A \) of the injective hull \( E(M) \), there exists a pure submodule \( P \) of \( M \) such that \( (A \cap M) \beta P \).

Proof: \((\Rightarrow)\) Let \( A \) be a direct summand of the injective hull \( E(M) \) of \( M \), then \((A \cap M) \) is a submodule of \( M \), since \( M \) is purely \( G \)-extending, then there exists a pure submodule \( P \) of \( M \) such that \((A \cap M) \beta P \).

\((\Leftarrow)\) Let \( A \) be a submodule of \( M \) and let \( B \) be a relative complement of \( A \) such that \( A \oplus B \) is essential in \( M \) [11]. Since \( M \) is essential in \( E(M) \), then \( A \oplus B \) is essential in \( E(M) \). Thus, \( E(A) \oplus E(B) = E(A \oplus B) = E(M) \) [10]. By hypothesis, there exists a pure submodule \( P \) of \( M \) such that \((E(A) \cap M) \beta P \). But \( A \) is essential in \( E(A) \). Therefore, \( A = (A \cap M) \leq (E(A) \cap M) \). But \( (A \cap M) = (A \cap M) \cap (E(A) \cap M) \leq (E(A) \cap M) \) and \( (A \cap M) = (A \cap M) \cap (E(A) \cap M) \leq (A \cap M) \). So, \( A = (A \cap M) \beta (E(A) \cap M) \). Since \( \beta \) is transitive, then \( A \beta P \). So \( M \) is purely \( G \)-extending module. ■

Proposition (2.3):
The following statements are equivalent for an an \( R \)-module \( M \):
(1) \( M \) is purely \( G \)–extending module.
(2) For each \( Y \) is a submodule of \( M \), there exists \( X \) a submodule of \( M \) and a pure submodule \( P \) of \( M \), such that \( X \leq^e Y \) and \( X \leq^e P \).

Proof: (1)\(\Rightarrow\)(2) Let \( Y \) be a submodule of \( M \). Then there exists a pure submodule \( P \) of \( M \) such that \( Y \beta P \), so \( Y \cap P \leq^e P \) and \( Y \cap P \leq^e Y \). The proof is complete put \( Y \cap P = Y \cap P \).

(2)\(\Rightarrow\)(1) Let \( Y \) be a submodule of \( M \). By (2), there exists a submodule \( X \) of \( M \) and a pure submodule \( P \) of \( M \) such that \( X \leq^e Y \) and \( X \leq^e P \). Now, since \( X \leq Y \cap P \leq Y \) and \( X \leq Y \cap P \leq P \) then \( Y \cap P \leq^e Y \) and \( Y \cap P \leq^e P \). So \( Y \beta P \) and so \( M \) is purely \( G \)–extending module. ■

Following [4], a direct sum of \( G \)-extending modules need not be \( G \)-extending module. Also, a direct sum of purely extending modules need not be purely extending module [5]. Here, we discuss when a direct sum of purely \( G \)-extending modules is a purely \( G \)-extending.
Recall that a submodule $N$ of an $R$-module $M$ is fully invariant if $f(N) \subseteq N$ for each $R$-endomorphism $f$ of $M$ [12]. $M$ is called Duo if every submodule of $M$ is fully invariant [13].

**Proposition (2.4)**

Let $M_i$ be purely $G$-extending $R$-module for each $i \in I$ such that every closed submodule of $M=\bigoplus_{i \in I} M_i$ is fully invariant, then $M=\bigoplus_{i \in I} M_i$ is purely $G$-extending module.

**Proof:** Let $K$ be a closed submodule of $M$ and let $\pi_i: M \rightarrow M_i$ be the natural projection on $M_i$ for each $i \in I$. Let $x \in K$, so $x = \sum_{i \in I} m_i$, where $m_i \in M_i$ and hence $\pi_i(x) = m_i$. Now, since $K$ is closed submodule of, then by hypothesis, $K$ is fully invariant and hence $\pi_i(K) \subseteq K \cap M_i$. So $\pi_i(x) = m_i \in K \cap M_i$ and hence $x \in \bigoplus_{i \in I} (K \cap M_i)$. Thus $K \subseteq \bigoplus_{i \in I} (K \cap M_i)$. Also, $\bigoplus_{i \in I} (K \cap M_i) \subseteq K$ and $\bigoplus_{i \in I} (K \cap M_i) = K$. Since $(K \cap M_i) \subseteq M_i$ and by purely $G$-extending property of $M_i$, then there is a pure submodule $P_i$ of $M_i$ such that $(K \cap M_i)\beta(P_i)$, $\forall i \in I$.

Now, since $P_i$ is a pure submodule of $M_i$, $\forall i \in I$, then $\bigoplus_{i \in I} P_i$ is a pure submodule in $M = \bigoplus_{i \in I} M_i$ [8]. So, $K = \bigoplus_{i \in I} (K \cap M_i)\beta(\bigoplus_{i \in I} P_i)$ [9]. Thus, $M$ is purely $G$-extending module.

**Corollary (2.5):**

Let $M = M_1 \oplus M_2$ be a duo module such that $M_1$ and $M_2$ are purely $G$-extending modules. Then $M$ is a purely $G$-extending. ■

By the same argument of the proof proposition (2.4), one can get the following result. Firstly, recall that an $R$-module $M$ is distributive if for all submodules $K, L$ and $N$ of $M$, $K \cap (L + N) = (K \cap L) + (K \cap N)$[14].

**Proposition (2.6)**

Let $M = M_1 \oplus M_2$ be a distributive module such that $M_1$ and $M_2$ are purely $G$-extending modules. Then $M$ is a purely $G$-extending.

**Proof:** Let $A$ is a submodule of $M = M_1 \oplus M_2$ since $M$ is a distributive module so $A = (A \cap M) = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. But $M_1$ and $M_2$ are purely $G$-extending, then there are a pure submodule $P_1$ of $M_1$ such that $(A \cap M_1)\beta(P_1)$ and pure submodule $P_2$ of $M_2$ such that $(A \cap M_2)\beta(P_2)$. So, $A = ((A \cap M_1) \oplus (A \cap M_2))\beta(P_1 \oplus P_2)$ by [9] and by [8] $(P_1 \oplus P_2)$ is a pure submodule of $M = M_1 \oplus M_2$. Thus, $M$ is a purely $G$-extending. ■

**Proposition (2.7):**

Let $M$ and $N$ be purely $G$-extending $R$-modules such that $ann(M) + ann(N) = R$. Then $M \oplus N$ is a purely $G$-extending module.

**Proof:** Let $A(\neq 0)$ be a submodule of $M \oplus N$. Since $ann(M) + ann(N) = R$, then $A = C \oplus D$, where $C$ is a submodule of $M$ and $D$ is a submodule of $N$[15]. Since $A(\neq 0)$ then $C(\neq 0)$ or $D(\neq 0)$. If $C \neq 0$ and $D = 0$, then $A = C$ is a submodule of $M$. But $M$ is purely $G$-extending and hence there is a pure submodule $H$ of $M$ such that $A\beta H$. Since $M$ is a direct summand of $M \oplus N$, then $M$ is a pure submodule of $M \oplus N$, (by [16]), then $H$ pure submodule of $M \oplus N$. Thus $M \oplus N$ is a purely $G$-extending module. By the similar way if $C = 0$ and $D \neq 0$, then $M \oplus N$ is a purely $G$-extending module. If $C(\neq 0)$ and $D(\neq 0)$, since $M$ and $N$ are purely $G$-extending modules, then there is a pure submodule $H$ of $M$ such that $C\beta H$, and there is a pure submodule $P$ of $N$ such that $D\beta P$. But $(H \oplus P)$ is a pure submodule of $M \oplus N$ [8] and by [9], $(C \oplus D)\beta(H \oplus P)$. Therefore, $M \oplus N$ is a purely $G$-extending module. ■
References
مقاسات التوسع التقنية من النمط -

سعد عبد الكاظم الساعدي
إقبال احمد عمر
قسم الرياضيات / كلية العلوم / الجامعة المستنصرية

استلم البحث في: ٤ آذار ٢٠١٥، قبل البحث في: ١٣ نيسان ٢٠١٥

الخلاصة

في هذا البحث، تم عرض ودراسة صنف من المقاسات التقنية. تم تعميم فعلي لكل من صنف مقاسات التوسع التقنية - من القيم M بأنه توسع إذا كان كل مقاس جزئي من X. ومقاسات التوسع التقنية. تم تعميم مقاسات التوسع التقنية - من القيم M بأنه توسع تقني من النقية من القيم M. ومقاسات التوسع التقنية. وكذلك تم تعميم فعلي لكل من صنف مقاسات التوسع التقنية من النمط - من القيم M. ومقاسات التوسع التقنية. أظهرت هذه الخاصية متحققة لمقاسات التوسع التقنية من النمط - من القيم M. ومقاسات التوسع التقنية.

المفتاحية: مقاسات التوسع، مقاسات التوسع التقنية، مقاسات التوسع من النمط -