W-Closed Submodule and Related Concepts

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Abstract

Let R be a commutative ring with identity, and M be a left unital module. In this paper we introduce and study the concept w-closed submodules, that is stronger form of the concept of closed submodules, where a submodule K of a module M is called w-closed in M, "if it has no proper weak essential extension in M", that is if there exists a submodule L of M with K is weak essential submodule of L then K=L. Some basic properties, examples of w-closed submodules are investigated, and some relationships between w-closed submodules and other related modules are studied. Furthermore, modules with chain condition on w-closed submodules are studied.

Keywords: Closed submodules, Weak essential submodules, W-closed submodules, completely essential modules, y-closed submodules, Minimal semi-prime submodules.
Introduction

In this note, we shall assume that all rings are commutative with unity and all modules are unital left modules, and all R-modules under study contains semi-prime submodules. "A submodule L of a module M is called closed in M provided that L has no proper essential extension in M [1]", "where a non-zero submodule N of M is called essential if N \cap E \neq (0) for all non-zero submodule E of M [1]", "and a non-zero submodule N of M is called weak essential if N \cap S \neq (0) \forall \text{ non-zero semi-prime submodule S of M} [2]". "Equivalently, a submodule N of a module M is called weak essential if whenever N \cap S \neq (0), then S=(0) for every semi-prime submodule S of M [3]", "where a submodule S of a module M is called semi-prime if for each r \in R \text{ and } y \in M \text{ with } r^k y \in S, k \in \mathbb{Z}^+ \text{ then } ry \in S [4]". "Equivalently if \( r^2 y \in S, \text{ then } ry \in S [5]". In this paper, "we introduce the concept of w-closed submodule "which is stronger than the concept of closed submodule" , where a submodule K of an R-module M is called w-closed "if K has no proper weak essential extension in M". That is if K is weak essential in L, where L is a submodule of M, then K=L. A module M is called chain if for each submodules E and D of M either E \subseteq D or D \subseteq E [6]. An R-module M is called fully semi-prime, if every proper submodule of M is semi-prime submodule [3]. A semi-prime radical of a module M denoted by Srad \( M \), and it is the intersection of all semi-prime submodule of M [3]. A submodule N of a module M is called y-closed submodule in M, if \( M \) is a non-singular module [1], "where an R-module M is called non-singular if \( \mathbb{Z}(M) = \{ x \in M : \text{ann}(x) \text{ is essential ideal in } R \} = (0) [3]". A module M is called multiplication module, if every submodule N of M is equal IM. i.e N=IM for some ideal I of R [7].

Basic Properties of W-Closed Submodules

"In this section, we introduce the definition of" w-closed submodule, and we will give basic properties, examples of w-closed submodule.

Definition (2.1)

A submodule K of a module M is called w-closed in M, "if K has no proper weak essential extension in M". That is if there exists submodule L of M with K "is a weak essential submodule of L", then K=L . An ideal J of R is called w-closed, if it is w-closed R-submodule.

Remark (2.2)

Every w-closed submodule in a module M is a closed submodule in M, but the converse is not true in general.

proof

Let K be a w-closed submodule in M and L is a submodule in M with K is essential in L, then by [2] K is weak essential in L. But K is w-closed in M, thus K=L. Hence K is closed submodule in M. For the converse, we give the following example:

Example(2.3)

Let M=\( Z_{24} \) as a Z-module, and K = \( \langle 3 \rangle \) is closed submodule in \( Z_{24} \), since K is a direct summand of the Z-module \( Z_{24} \), but K is not w-closed submodule in \( Z_{24} \) because K is weak essential submodule in \( Z_{24} \).

Proposition (2.4)

If M is a module, and E is a submodule of M such that E is weak essential and w-closed in M, then E=M.
Proof
Follows from definition of w-closed submodule.

Remark (2.5)
(1) Every module M is a w-closed submodule in itself.

(2) The trivial submodule <0> may not be w-closed submodule of an R-module M, for example : $M = Z_2$ as a Z-module, $K = \langle 0 \rangle$ is not w-closed submodule in M.

Proposition(2.6)
If M is a module, and let U be a non-zero submodule of M, then there exists a w-closed submodule T in M with U is weak essential in T.

Proof
Let $\mathcal{A} = \{ Q : Q \text{ is a submodule of } M \text{ such that } U \text{ is weak essential in } Q \}$. Clearly $\mathcal{A}$ is a non-empty. $\mathcal{A}$ has maximal element say T "by Zorn's lemma". "To prove that" T is a w-closed submodule in M. Assume that there exists a submodule L of M with T weak essential in L. Since U is weak essential in T and T is weak essential in L so by [3, prop (1.4)]. U is weak essential in L. But this is a contradicts the maximality of T. Thus T=L. Hence T is w-closed submodule in M, with U is weak essential in T.

The following remark shows that w-closed property is not hereditary property.

Remark(2.7)
If $Q_1$ and $Q_2$ are submodules of an R-module M with $Q_1$ is a submodule of $Q_2$, and $Q_2$ is a w-closed submodule in M then $Q_1$ need not to be w-closed submodule in M. For example: $M=Z$ the Z-module, M is a w-closed submodule of M, and $2Z$ is a submodule of M is not w-closed submodule in M, since $2Z$ has a proper weak essential extension.

The converse of remark (2.7) is not true. That is if $Q_1$ is w-closed in M, then $Q_2$ need not to be w-closed in M. As the next example explain:

Example(2.8)
Take the Z-module Z and $N_1 = \langle 0 \rangle$, $N_2 = 2Z$ are Z-submodules of Z we notes that $N_1$ is w-closed submodule in Z. But $N_2$ is not w-closed submodule in Z.

The following propositions show that the transitive property for w-closed submodule hold under certain conditions.

Proposition (2.9)
If E and D are submodules of a module M, provided that D contained in any weak essential extensions of E, and E is a w-closed submodule in D and D is a w-closed submodule in M, then E is a w-closed submodule in M.

Proof
Assume that K is a submodule of M such that E is weak essential in K. By hypothesis D is a submodule of K. Since E "is weak essential in K and E is a submodule of D" then by [2, Rem(1.5)(2)] we get D is weak essential in K. But D is w-closed submodule in M, then D=K. That is E weak essential in D. But E is w-closed submodule in D, so E=D. Hence E is a w-closed submodule in M.
Proposition (2.10)

If \( N_1 \) and \( N_2 \) are submodules of a module \( M \), provided that \( N_2 \) is containing any weak essential extensions of \( N_1 \), and \( N_1 \) is a w-closed submodule in \( N_2 \) and \( N_2 \) is a w-closed submodule in \( M \), then \( N_1 \) is a w-closed submodule in \( M \).

Proof

Assume that \( U \subseteq M \) with \( N_1 \) is weak essential submodule in \( U \), then by hybothesis we get \( U \) is a submodule in \( N_2 \). Since \( N_1 \) is a w-closed in \( N_2 \), then \( N_1 = U \). Thus \( N_1 \) is a w-closed submodule in \( M \).

Proposition (2.11)

If \( M \) is a chained module, and \( E, D \) are submodules of \( M \) with \( E \subseteq D \), and \( E \subseteq \_M \) and \( D \subseteq \_M \), then \( E \subseteq \_M \).

Proof

Let \( K \) be a submodule of \( M \) with \( E \) is weak essential in \( K \). Since \( M \) is chained module, then either \( K \) is a submodule in \( D \) or \( D \) is a submodule in \( K \). If \( K \) is a submodule in \( D \), and since \( E \) is a w-closed submodule in \( D \), then \( E = K \). Hence \( E \) is a w-closed submodule in \( M \). If \( D \) is a submodule in \( K \), and since \( E \) is weak essential in \( K \), then by [2, Rem(1.5)(2)] \( D \) is a weak essential submodule in \( K \). But \( D \) is a w-closed submodule in \( M \), hence \( D = K \). Thus, \( E \) is a weak essential submodule in \( D \). But \( E \) is a w-closed submodule in \( D \), then \( E = D \). Hence \( E \) is a w-closed submodule in \( M \).

Before we give the next proposition, we introduce the following definition.

"Definition (2.12)

A module \( M \) is called completely essential if every non zero weak essential submodule of \( M \) is an essential submodule of \( M \)."

Completely essential in [3] is called fully essential.

The following proposition show that closed submodules and w-closed submodules are equivalents under certain conditions.

Proposition (2.13)

"If \( M \) is a module, and \( E \) be a non zero submodule of \( M \" such that every weak essential extensions of \( E \) is a completely essential, then \( E \) is a closed submodule in \( M \) if and only if \( E \) is a w-closed submodule in \( M \).

Proof

Let \( E \) be a non zero closed submodule in \( M \), and \( U \) be a submodule of \( M \) such that \( E \) is a weak essential in \( U \). By hypothesis \( U \) is a completely essential, therefore \( E \) is an essential submodule in \( U \). But \( E \) is a closed submodule in \( M \), then \( E = U \). That is \( E \) is a w-closed submodule.

The converse is direct.

Proposition (2.14)

If \( M \) is a fully semi-prime module, and \( E \) be a non zero submodule of \( M \), then \( E \) is a closed submodule in \( M \) if and only if \( E \) is a w-closed submodule in \( M \).
**Proof**
Assume that $E$ is a non zero closed submodule in $M$, and $U$ is a submodule of $M$ such that $E$ is a weak essential submodule in $U$. Then by [3, Cor(2.5)] $E$ is an essential submodule in $U$. But $E$ is a non-zero closed submodule in $M$, hence $E=U$. That is $E$ is a $w$-closed submodule in $M$.

The converse is direct.

**Corollary (2.15)**
If $M$ is a uniform module, and $E$ be a non zero submodule of $M$, then $E$ is a closed submodule in $M$ if and only if $E$ is a $w$-closed submodule in $M$.

**Proof**
Assume that $E$ is a closed submodule in $M$ and let $E$ a weak essential in $U$ where $U$ is a submodule of $M$, then $U$ is a uniform. Hence by [3,prop(2.7)] $U$ is a completely essential. Thus $E$ is an essential in $U$. But $E$ is a closed, then $E=U$. Thus $E$ is a $w$-closed in $M$.

The converse is direct.

The following propositions show that the transitive property for $w$-closed submodules hold under conditions fully semi-prime and completely essential.

**Proposition (2.16)**
Let $M$ be a module, and $E, D$ are non-zero submodules of $M$ such that $E \leq D$ and every weak essential extensions of $E$ is a completely essential submodule of $M$. If $E \leq_w D$ and $D \leq_w M$, then $E \leq_w M$.

**Proof**
Since $E \leq_w D$ and $D \leq_w M$. Then by remark(2.2), we get $E$ is a closed submodule in $D$ and $D$ is a closed submodule in $M$. Then by [1,prop(1.5), P.18] "we get $E$ is a closed submodule in $M"$, then by prop(2.13), $E \leq_w M$.

**Proposition (2.17)**
Let $M$ be a fully semi-prime module, and let $E$ be a non-zero $w$-closed submodule in $D$ and $D$ is a $w$-closed submodule in $M$. Then $E$ is a $w$-closed submodule in $M$.

**Proof**
Since $E$ is a $w$-closed submodule in $D$ and $D$ is a $w$-closed submodule in $M$, then by remark(2.2), $E$ is a closed submodule in $D$ and $D$ is a closed submodule in $M$. Hence by [1,prop(1.5), P.18] we get $E$ is a closed submodule in $M$. Thus by prop(2.14), $E$ is a $w$-closed submodule in $M$.

**Remark (2.18)**
The intersection of two $w$-closed submodule need not to be $w$-closed submodule as the following example shows:

In the $Z$-module $Z_8 \oplus Z_2$, the submodules $N = \langle (0, 1) \rangle$ and $K = \langle (4, 1) \rangle$ are $w$-closed submodule in $Z_8 \oplus Z_2$, but $N \cap K = \langle (0, 0) \rangle$ is not $w$-closed submodule in $Z_8 \oplus Z_2$.

The following results give more basic properties of $w$-closed submodules.
**Proposition (2.19)**

If every submodule of a module $M$ is $w$-closed, then every submodule of $M$ is a direct summand. Provided that $M$ is a semi simple.

**Proof**

Since every submodule of $M$ is $w$-closed, then every submodule of $M$ is a closed. Hence by [8, Exc(6-c), P.139] "every submodule of $M$ is a direct summand of $M$".

The following corollary is a direct consequence of proposition(2.19).

**Corollary (2.20)**

If every submodule of a module $M$ is a $w$-closed, then $M$ is a semi-simple.

**Proposition(2.21)**

If $E$ and $D$ are submodules of a module $M$ with $E \leq D$, and $E \leq_w M$, then $E \leq_w D$.

**Proof**

Let $F \leq D$, then $F \leq M$, and $E$ is a weak essential submodule of $F$. But $E \leq_w M$, then $E=F$. Hence $E \leq D$.

As a direct application of proposition(2.21) we get the following results.

**Corollary (2.22)**

If $E$ and $D$ are submodules of a module $M$ with $E \cap D$ is a $w$-closed submodule in $M$, then $E \cap D$ is a $w$-closed submodule in $E$ and $D$.

**Corollary (2.23)**

If $M$ is a module, and $E$, $U$ are $w$-closed submodules in $M$, then $E$ and $U$ are $w$-closed submodules in $E + U$.

**Corollary(2.24)**

If $M$ is an $R$-module, and $E$ is a $w$-closed submodule in $M$, then $E$ is a $w$-closed submodule in $\sqrt{E}$.

**Proof**

Since $E \leq \sqrt{E} \leq M$, and $E$ is a $w$-closed submodule in $M$ then by proposition(2.21), $E$ is a $w$-closed submodule in $\sqrt{E}$.

**Remark (2.25)**

A direct summand of a module $M$ is not necessary $w$-closed submodule in $M$, as the following example show:

Let $M=Z_{24}$ as a $Z$-module, where $Z_{24} = \langle 3 \rangle \oplus \langle 8 \rangle$, the direct summand $\langle 3 \rangle$ is not $w$-closed submodule in $Z_{24}$. Since $\langle 3 \rangle$ is a weak essential in $Z_{24}$.

**Proposition(2.26)**

Let $X = X_1 \oplus X_2$ be a module, where $X_1$ and $X_2$ are submodules of $X$, and let $E$ be a non zero $w$-closed submodule in $X_1$ and $D$ is a non zero $w$-closed submodule in $X_2$ such that $\text{ann } X_1 + \text{ann } X_2=R$, and all weak-essential extensions of $E \oplus D$ are completely essential submodule of $X_1 \oplus X_2$. Then $E \oplus D$ is a $w$-closed submodule in $X_1 \oplus X_2$. 

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Proof
Let $S \leq X$ with $E \oplus D$ "is a weak essential submodule in S". Since $S$ is a submodule of $X$ and $\text{ann } X_1 + \text{ann } X_2 = R$, then by [9, prop(4.2)], $S = S_1 \oplus S_2$, where $S_1$ is a submodule of $X_1$ and $S_2$ is a submodule of $X_2$. Thus $E \oplus D$ is a weak essential submodule in $S_1 \oplus S_2$. But by hypothesis $S$ is a completely essential, therefore $E \oplus D$ is an essential submodule in $S = S_1 \oplus S_2$, thus by [10, prop(5.20)] we are, "E is an essential submodule in $S_1$ and D is an essential submodule in $S_2$". Since both $E$ and $D$ are $w$-closed, it is a clear that $E$ and $D$ are closed submodules in $S_1$ and $S_2$ respectively. Then $E=S_1$ and $D=S_2$, thus $E \oplus D = S_1 \oplus S_2$. That is $E \oplus D$ is a $w$-closed submodule in $X$.

Proposition (2.27)
Let $X = X_1 \oplus X_2$ be a module, where $X_1$ and $X_2$ are submodules of $X$ such that $\text{ann } X_1 + \text{ann } X_2 = R$ and all submodules of $X$ are completely essential submodule of $X$. If $E$ and $D$ are non zero submodules of $X_1$ and $X_2$ respectively, then $E \oplus D$ is a $w$-closed submodule in $X$ if and only if $E$ is a $w$-closed submodule in $X_1$ and $D$ is a $w$-closed submodule in $X_2$.

Proof
($\Leftarrow$) Suppose that $E \oplus D$ "is weak essential submodule of $K$", "where K is a submodule of M". Hence by [1, prop(4.2)] $K = K_1 \oplus K_2$ where $K_1$ is a submodule of $X_1$ and $K_2$ is a submodule of $X_2$. Thus $E \oplus D$ is weak essential submodule in $K_1 \oplus K_2$. But $K_1 \oplus K_2$ is a completely "essential submodule of " $X$, then $E \oplus D$ "is an essential submodule of " $K_1 \oplus K_2$. Hence by [10, prop(5.20), P.15] we get "E is an essential submodule in $K_1$ and D is an essential submodule in $K_2". But by [2] every essential submodule is a weak essential. Hence E "is a weak essential submodule in $K_1"$ and D is a weak essential submodule in $K_2$. But E and D are $w$-closed submodules of $X$, then $E=K_1$ and $D=K_2$. Thus $E \oplus D = K_1 \oplus K_2$. That is $E \oplus D$ is a $w$-closed submodule in $X$.

($\Rightarrow$) Assume that E "is a weak essential submodule in $L$" where $L$ is a submodule of $X$, we have $D$ is a weak essential submodule in $D$. But by hypothesis all submodules of $X$ are completely essential, then E is an essential submodule in $L$ and $D$ is an essential submodule in $D$. Hence by [10, prop(5.20), P.15], we have $E \oplus D$ is an essential submodule in $L \oplus D$, which implies that $E \oplus D$ is a weak essential submodule in $L \oplus D$. Hence $E \oplus D = L \oplus D$. That is $E = L$, implies that E is a $w$-closed submodule in $X_1$.

In similar way we can prove that $D$ is $w$-closed submodule in $X_2$.

It is well-known that a fully semi-prime module is a completely essential [3, cor(2.6)]. So we have the following result.

Corollary (2.28)
If $X = X_1 \oplus X_2$ is a module, where $X_1$ and $X_2$ are submodules of $X$ with $\text{ann } X_1 + \text{ann } X_2 = R$ and all submodules of $X$ are fully-semi-prime. If $E$, $D$ are submodules of $X_1$ and $X_2$ respectively, then $E \oplus D$ is a $w$-closed submodule in $X$ if and only if $E$ is a $w$-closed submodule in $X_1$ and $D$ is a $w$-closed submodule in $X_2$.

The following remark shows that $w$-closed property is not algebraic property.

Remark (2.29)
If $M$ is a module, and $X$ is a $w$-closed submodule of $M$, and $Y$ is submodule of $M$ such that $X \equiv Y$, then it is not necessary that $Y$ is a $w$-closed submodule in $M$, as the following example
shows:- The $Z$-module $Z$ is a w-closed in itself and $Z \cong 3Z$, but $3Z$ as a $Z$-module is not a w-closed submodule in $Z$, since $3Z$ "is a weak-essential submodule of $Z".\]

We introduce the following lemma, before we give the next proposition.

**Lemma(2.30)**

Let $f \in Hom(M_1,M_2)$ be module an epimorphism with $Ker f \leq Srad(M_1)$, if $E \leq_{weak} M_2$. Then $f^{-1}(E) \leq_{weak} M_1$.

**Proof**

Assume that $E \leq_{weak} M_2$, and $f^{-1}(E) \cap S = (0)$ where $S$ is a semi-prime submodule of $M_1$. But $Ker f \leq Srad(M_1) \leq S$ for all semi-prime submodule $S$ of $M_1$, hence by [5, prop(2.1)(A)] $f(S)$ is a semi-prime submodule of $M_2$. That is $E \cap f(S) = (0)$, but $E$ "is a weak essential submodule of $M_2"$, then $f(S) = (0)$. Implies that $S \leq Ker f \leq f^{-1}(E)$, and hence $f^{-1}(E) \cap S = (0)$ implies that $S = (0)$. Then $f^{-1}(E) \leq_{weak} M_1$.

**Proposition(2.31)**

Let $g: M_1 \rightarrow M_2$ be a module epimorphism, and let $E$ be a submodule of $M_1$ such that $Ker g \leq Srad(M_1) \cap E$. If $E$ is a w-closed submodule in $M_1$ then $g(E)$ is a w-closed submodule in $M_2$.

**Proof**

Suppose that $E$ is a w-closed submodule in $M_1$, and let $g(E)$ "is a weak essential submodule of $L"$, where $L$ is a submodule of $M_2$. Since $Ker g \leq Srad(M_1) \cap E$. Hence by lemma(2.30), we get $g^{-1}(g(E))$ is a weak essential submodule in $g^{-1}(L)$, where $g^{-1}(L)$ is a submodule of $M_1$, but $Ker g \leq E$, then $g^{-1}(g(E)) = E$, i.e $E$ is a weak essential in $g^{-1}(L)$. But $E$ is a w-closed submodule in $M_1$, then $E=g^{-1}(L)$, and since $g$ is an epimorphism so, $g(E) = L$. Hence $g(E)$ is a w-closed submodule in $M_2$.

As a direct consequence of proposition(2.31) we get the following corollary.

**Corollary(2.32) :** If $E$ and $D$ are submodules of a module $M$ with $E \leq srad(M) \cap D$. If $D$ is a w-closed submodule in $M$, then $E$ is a w-closed submodule in $M$.

The following proposition gives a relation between y-closed submodule and w-closed submodule in the class of a fully semi-prime module.

**Proposition (2.33)**

Let $M$ be a fully semi-prime module. Then every non zero y-closed submodule is a w-closed submodule.

**Proof**

Let $E$ be a non zero y-closed submodule in $M$, then by [11], every y-closed submodule is a closed. Hence $E$ is a closed, then by proposition(2.14), $E$ is a w-closed submodule in $M$.

"The following proposition shows that in the class of non-singular modules", the class of w-closed submodules is contained in the class of y-closed submodules.

**Proposition (2.34)**

If $M$ is a non singular module and $E$ is a w-closed submodule of $M$, then $E$ is a y-closed submodule of $M$. 

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Proof

Let \( E \) be a \( w \)-closed submodule in \( M \) then \( E \) "is a closed submodule in \( M \)". but \( M \) is a non-singular \( R \)-module, then by [11, prop(2.1)(2)] \( E \) is a \( y \)-closed submodule in \( M \).

The following proposition shows that in the class of non-singular and fully semi-prime \( R \)-module, \( w \)-closed submodule, \( y \)-closed submodule and closed submodule are equivalent:

**Proposition (2.35)**

Let \( M \) be a fully semi-prime and non-singular module, "and \( E \) be a non zero submodule of \( M \). Then the following statements are equivalent" :

1- \( E \) is a \( y \)-closed submodule.
2- \( E \) is a closed submodule.
3- \( E \) is a \( w \)-closed submodule.

**Proof**

\((1) \Rightarrow (2)\) Follows by [11].

\((2) \Rightarrow (3)\) Follows by proposition(2.14).

\((3) \Rightarrow (1)\) Follows by proposition(2.34).

3. \( W \)-closed submodule in multiplication modules

In this section, we establish some relationships between \( w \)-closed submodule and multiplication modules.

"First we introduce the following definition".

**Definition(3.1)**

A non-zero semi-prime submodule \( E \) of a module \( M \) is called minimal semi-prime submodule of \( M \), if whenever \( S \) "is a non zero semi-prime submodule of \( M \) such that" \( S \leq E \), then \( S=E \). That is by minimal semi-prime submodule \( E \) of \( M \) we mean a semi-prime submodule which is a minimal in the collection of semi-prime submodules of \( M \). If \( A \) is a proper ideal of \( R \), then a semi-prime ideal \( B \) is called a minimal semi-prime ideal of \( A \) provided that \( A \leq B \) and \( \frac{B}{A} \) is minimal semi-prime ideal of a ring \( \frac{R}{A} \).

**Remark(3.2)**

In multiplication module since \( \text{ann}(M) \neq R \) it follows that by [12, Th(2.5)], there exists a minimal ideal \( P \) of \( R \) such that \( \text{ann}(M) \leq P, \text{and } M \neq PM \). But by [13, prop(2.5), P.36] \( PM \) is a semi-prime submodule of \( M \).

Then from definition(3.1) we get the following facts:

(a) \( E \) is a minimal semi-prime submodule of \( M \) if and only if there exists a minimal semi-prime ideal \( A \), with \( \text{ann}(M) \leq A \) such that \( E = AM \neq M \).

(b) Every semi-prime submodule of \( M \) contains a minimal semi-prime submodule.
Lemma (3.3)

If M is a faithful and multiplication module, and E be a non zero semi-prime submodule of M. If E is not minimal semi-prime, then E "is a weak-essential submodule of M".

Proof

Since M is a multiplication, and E is a semi-prime submodule of M, then by [13, prop(2.5), P.36] \( \exists \text{ a "semi-prime ideal K of R" with } (0) = \text{ann} M \leq K \) and \( E = KM. " \) Let S be a non-zero semi-prime submodule of M" such that \( E \cap S = (0) \). But E is not minimal semi-prime, then by remark(3.2)(b) every semi-prime submodule of M contain a minimal semi-prime submodule say \( E_1 \leq E \). Hence by remark(3.2)(a), there exists a minimal semi-prime ideal \( K_1 \) of R such that \( \text{ann}(M) \leq K_1 \) and \( E_1 = K_1 M \neq M \). \( K \cap [S: M] M = K M \cap [S: M] M = E \cap S = (0) = (0) \). But M is faithful, then \( K \cap [S: M] = (0) \), which implies that \( K \cap [S: M] \leq K_1 \), that is either \( K \leq K_1 \) or \( [S: M] \leq K_1 \). If \( K \leq K_1 \), then \( K M \leq K_1 M \), implies that \( E \leq E_1 \) which is a contradiction. Thus, \( [S: M] \leq K_1 \). That is \( [S: M] M \leq K_1 M \), implies that \( S \leq E_1 \leq E \) which is contradict the minimality of \( E_1 \). Thus \( E \cap S = (0) \) is not true. Thus \( E \cap S \neq (0) \), which implies that E is a weak essential submodule of M.

Proposition (3.4)

If M is a faithful and multiplication module, and E be a non-zero semi-prime submodule and w-closed submodule of M, then E is a minimal semi-prime submodule of M.

Proof

Suppose that E is not minimal semi-prime submodule of M, then by lemma(3.3), E "is a weak essential submodule of M". But E is a w-closed submodule in M, then E=M. On the other hand E is a semi-prime submodule of M, that E must be a proper submodule of M, so we get contradiction. Hence E must be a minimal "semi-prime submodule of M".

Proposition (3.5)

Let M be a non zero multiplication module with only one non zero maximal submodule E. Then E can not be w-closed submodule in M.

Proof

Assume that E is a w-closed submodule in M, then by [3, prop(2.20)] E "is a weak essential submodule of M". Hence E=M. "But this contradict the maximality of E". Therefore E is not W-closed submodule in M.

"Recall that for any module M and any ideals I and J of R if I is a semi-prime ideal of J then IM is a semi-prime submodule of JM this is called condition(\(*)\) in [3]."

Proposition(3.6)

Let M be a faithful and multiplication module such that M satisfies condition(\(*)\), if L is a w-closed ideal in K then LM is a w-closed submodule in KM.

Proof

Suppose that L is a w-closed ideal in K, and LM is a weak essential submodule of T where T is a submodule of KM, we have to show that LM=T. Since M is a multiplication module, then \( T = PM \) for some ideal P of R with \( P \leq K \). That is LM "is a weak essential submodule of PM", and since M is faithful and satisfies condition(\(*)\) then by [3,prop(2.17)], we have L is a weak
essential ideal in P and $P \leq K$. But L is a w-closed ideal in K, then $L = P$. That is $LM = PM = T$.
Hence LM is a w-closed submodule in KM.

The following proposition gives the converse of proposition (3.6).

**Proposition (3.7)**

If M is a finitely generated, faithful and multiplication module, and LM is a w-closed submodule in KM, then L is a w-closed ideal in K.

**Proof**

Suppose that LM is a w-closed submodule in KM, where L and K are ideals in R, and let L is a weak essential ideal in U where U is an ideal of K. "Since M is finitely generated faithful and multiplication", then by [3, prop(2.18)] we have LM is a weak essential in UM which is a submodule of KM. But LM is a w-closed submodule in KM, then $LM = UM$. Hence by [12, Th,(3.1)], $L = U$. Then L is a w-closed ideal in K.

From proposition (3.6) and proposition (3.7) we get the following corollary.

**Corollary (3.8)**

"If M is a finitely generated faithful and multiplication module which satisfies condition $(\ast)$", then L is a w-closed ideal in K if and only if LM is a w-closed submodule in KM.

**Theorem (3.9)**

If M is a finitely generated faithful and multiplication module, and let E be a submodule of M, such that $M$ satisfies condition $(\ast)$, "then the following statements are equivalent" :

1- E is a w-closed submodule in M.
2- $[E : R M]$ is a w-closed ideal in R.
3- E=PM for some w-closed ideal P in R.

**Proof**

(1) $\Rightarrow$ (2) Suppose that E is a w-closed submodule in M. Since M is a multiplication, then by [7] $E = [E : R M] M$. Put $[E : R M] = P$, then we have PM=E is a w-closed submodule in M. Hence by cor(3.8), P is a w-closed ideal in R. That is $[E : R M]$ is a w-closed ideal in R.

(2) $\Rightarrow$ (3) : Suppose that $[E : R M]$ is a w-closed ideal in R. Then $E = [E : R M] M$ since M is multiplication, i.e E=PM where $P = [E : R M]$ is a w-closed ideal in R.

PM=E is a w-closed submodule in RM=M.

**4- Chain conditions on w-closed submodules**

We start this section by introducing the definitions of a modules that have ascending (descending) chain condition on w-closed submodules.

**Definition (4.1)**

A module M is said to have the ascending chain condition on w-closed
submodule (briefly acc on w-closed submodules), if every ascending chain $E_1 \subseteq E_2 \subseteq \ldots$ of w-closed submodule in M is finite. That is $\exists \ m \in \mathbb{Z}_+$ such that $E_n = E_m$ for all $n \geq m$.

Definition (4.2)

A module M is said to have the descending chain condition on w-closed submodule (briefly dcc on w-closed submodules), if every descending chain $E_1 \supseteq E_2 \supseteq \ldots$ of w-closed submodule in M is finite. That is $\exists \ m \in \mathbb{Z}_+$ such that $E_n = E_m$ for all $n \geq m$.

Remarks (4.3)

1- $\mathbb{Z}p^\infty$ as a $\mathbb{Z}$-module satisfies dcc on w-closed submodules, but $\mathbb{Z}p^\infty$ as a Z-modules does not satisfies acc on w-closed submodules because $\mathbb{Z}p^\infty$ is an artinian but not noetherian.

2- Z as Z-module satisfies (acc) on w-closed submodules, but does not satisfies (dcc) on w-closed submodules because Z as a Z-module is a noetherian but not artinian.

Proposition (4.4)

If M is a module and satisfies (dcc) on closed submodules, then M satisfies (dcc) on w-closed submodules.

Proof

Let $E_1 \supseteq E_2 \supseteq \ldots$ "be a descending chain" of w-closed submodules of M. But by remark(2.2) every w-closed submodule is closed, then $E_i$ is a closed submodule for each $i=1,2, \ldots$. Since M satisfies (dcc) on closed submodule, then $\exists \ m \in \mathbb{Z}_+$ such that $E_n = E_m$ for each $n \geq m$. Thus, M satisfies (dcc) on w-closed submodules.

The proof of the following proposition is similar to the proof of proposition (4.4) and hence is omitted.

Proposition (4.5)

If M is a module and satisfies (acc) on closed submodules, then M satisfies (acc) on w-closed submodules.

Since w-closed submodules and closed submodules are equivalent in the class of fully semi-prime modules by proposition (2.14), "we get the following results".

Proposition (4.6)

If M is a fully semi-prime module, then M satisfies (acc) on w-closed submodules if and only if M satisfies (acc) on closed submodules.

Proof

($\Rightarrow$) Let $E_1 \subseteq E_2 \subseteq \ldots$ "be ascending chain of closed submodules". Then by prop(2.14), $E_i$ is a w-closed submodule for each $i=1,2, \ldots$. But M satisfies (acc) on w-closed submodules, so $\exists \ m \in \mathbb{Z}_+$ such that $E_n = E_m$ for all $n \geq m$. Thus M satisfies (acc) on closed submodules.

($\Leftarrow$) By proposition (4.5).

The proof of the following proposition is similar to proof of proposition (4.6).

Proposition (4.7)

Let M be a fully semi-prime module. "Then M satisfies (dcc) on closed submodules if and only if M satisfies (dcc)" on w-closed submodules.
Proposition (4.8)

If $M$ is a module, and $E_1 \subseteq E_2 \subseteq \ldots$ be an ascending chain of submodules such that each weak essential extension of $E_i$ is a completely essential for each $i=1,2,\ldots$, then $M$ satisfies (acc) on w-closed submodules if and only if $M$ satisfies (acc) on closed submodules.

Proof

($\Rightarrow$) Let $E_1 \subseteq E_2 \subseteq \ldots$ be an ascending chain of closed submodules. Then by prop (2.13), $E_i$ is a w-closed submodule for each $i=1,2,\ldots$. But $M$ satisfies (acc) on w-closed submodules, then there exists a non-zero integer $m$ such that $E_n = E_m$ for all $n \geq m$. Hence $M$ satisfies (acc) on closed submodules.

($\Leftarrow$) Follows by proposition (4.5).

The proof the following proposition is similar to proof of proposition (4.8).

Proposition (4.9)

If $M$ is a module, and $E_1 \supseteq E_2 \supseteq \ldots$ be a descending chain of submodules such that each weak essential extension of $E_i$ is a completely essential for each $i=1,2,\ldots$. Then $M$ satisfies (dcc) on w-closed submodules if and only if $M$ satisfies (dcc) on closed submodules.

Proposition (4.10)

If $M$ is a module, and $D$ be a submodule of $M$ such that $D \subseteq S_{rad}(M) \cap K$, where $K$ is any w-closed submodule in $M$. If $\frac{M}{D}$ satisfies (dcc) on w-closed submodules, then $M$ satisfies (dcc) on w-closed submodules.

Proof

Let $E_1 \supseteq E_2 \supseteq \ldots$ be a descending chain of w-closed submodules in $M$. Since $E_i$ is a w-closed submodule in $M$ for each $i=1,2,\ldots$, and $D \subseteq S_{rad}(M) \cap E_i$ then by Corollary (2.32), we have $\frac{E_i}{D}$ is a w-closed submodule in $\frac{M}{D}$ for each $i=1,2,\ldots$. Hence $\frac{E_1}{D} \supseteq \frac{E_2}{D} \supseteq \ldots$, is a descending chain of w-closed submodules in $\frac{M}{D}$. But $\frac{M}{D}$ satisfies (dcc) on w-closed submodules, so there exists a positive integer $m$ such that $\frac{E_n}{D} = \frac{E_m}{D}$ for each $n \geq m$. So, that $E_n = E_m$ for each $n \geq m$. Thus $M$ satisfies (dcc) on w-closed submodules.

Proposition (4.11)

If $M$ is a module, and $D$ be a submodule of $M$ such that $D \subseteq S_{rad}(M) \cap K$, where $K$ is any w-closed submodule in $M$. If $\frac{M}{D}$ satisfies (acc) on w-closed submodules, then $M$ satisfies (acc) on w-closed submodules.

Proof

Similar to proof of proposition (4.10).

Proposition (4.12)

If $X = X_1 \oplus X_2$ is a module, where $X_1$ and $X_2$ are submodules of $X$, provided that $\text{ann} X_1 + \text{ann} X_2 = \mathbb{R}$, and all weak essential extensions of $E_i \oplus X_2$ (or $X_1 \oplus E_i$...
are completely essential modules where $E_i$ is a non zero w-closed submodule in $X_1$ (or $X_2$) for each $i=1,2,\ldots$. If $X$ satisfies (dcc) on w-closed submodules, then $X_1$ (or $X_2$) satisfies (dcc) on non-zero w-closed submodules.

**Proof**

Let $E_1 \supseteq E_2 \supseteq \ldots$ "be a descending chain" of a non-zero w-closed submodules of $X_1$. If $X_2$ is equal to zero, then $X=X_1$ and this, implies that $X_1$ satisfies (dcc) on non-zero w-closed submodules. Otherwise, since $E_i$ is a non-zero w-closed submodule in $X_1$, and $X_2$ is a w-closed in $X_2$, so by proposition(2.26), $E_i \oplus X_2$ is a w-closed submodule in $X$ for each $i=1,2,\ldots$, "is a descending chain" of w-closed submodule in $X$. But $X$ satisfies (dcc) on w-closed submodules, then there exists a positive integer $m$ such that $E_n \oplus X_2 = E_m \oplus X_2$ for all $n \geq m$. Thus $E_n = E_m$ for all $n \geq m$. Thus $X_1$ satisfies (dcc) on w-closed submodule.

Similarly we can prove that $X_2$ satisfies (dcc) on w-closed submodule.

**Proposition(4.13)**

If $X = X_1 \oplus X_2$ is a module, where $X_1$ and $X_2$ are submodules of $X$, provided that $\text{ann } X_1 + \text{ann } X_2 = R$, and all weak essential extensions of $E_i \oplus X_2$ (or $X_1 \oplus E_i$) are completely essential modules where $E_i$ is a non zero w-closed submodule in $X_1$ (or $X_2$) for each $i=1,2,\ldots$. If $X$ satisfies (acc) on w-closed submodules, then $X_1$ (or $X_2$) satisfies (acc) on non-zero w-closed submodules.

**Proof**

Similarly as in proposition (4.12).

We end this section by the following propositions.

**Proposition(4.14)**

"If $M$ is a finitely generated,faithful and multiplication module, and $M$ satisfies condition(+)", then $M$ satisfies (dcc) on w-closed submodules, if and only if $R$ satisfies (dcc) on w-closed ideals.

**Proof**

$(\Rightarrow)$ Let $L_1 \supseteq L_2 \supseteq \ldots$, "be a descending chain" of w-closed ideals in $R$. Since $L_i$ is a w-closed ideal in $R$ for each $i=1,2,\ldots$. Then by cor(3.8) $L_i M$ is a w-closed submodule in $M$ for each $i=1,2,\ldots$, then $L_1 M \supseteq L_2 M \supseteq \ldots$, be a "descending chain" of w-closed submodules in $M$. But $M$ satisfies (dcc) on w-closed submodules, "so there exists a positive integer $m$ such that" $L_n M = L_m M$ for each $n \geq m$. But $M$ is a finitely generated, faithful and multiplication R-module, then by [12, Th(3.1)], $L_n = L_m$ for each $n \geq m$. Therefore $R$ satisfies (dcc) on w-closed ideals.

$(\Leftarrow)$ Let $E_1 \supseteq E_2 \supseteq \ldots$, be a descending chain of w-closed submodules in $M$. Since $M$ is multiplication module, then $E_i = L_i M$ for some ideal $L_i$ of $R$ for each $i=1,2,\ldots$. Since $E_i$ is a w-closed submodule in $M$ for each $i=1,2,\ldots$, so by cor(3.8), $L_i$ is a w-closed ideal in $R$ for each $i=1,2,\ldots$. But $M$ is a finitely generated, faithful and multiplication module, then by [12, Th(3.1)] we have $L_1 \supseteq L_2 \supseteq \ldots$, is a "descending chain" of w-closed ideals in $R$. But $R$ satisfies (dcc) on w-closed ideals, therefore, there exists a positive integer $m$ such that $L_n M = L_m M$ for each $n \geq m$, thus $E_n = E_m$ for each $n \geq m$.

The proof the following proposition is similar to the proof of prop(4.14), hence we omitted.
Proposition (4.15)

"If M is a finitely generated, faithful and multiplication module, and M satisfies condition (∗), then M satisfies (acc) on w-closed submodules, if and only if R satisfies (acc) on w-closed ideals.

References

1. Goodearl K.R. (1976), " Ring Theory, Nonsingular Rings and Modules ", Marcel, Dekker, New York,

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