Quasi-inner product spaces of quasi-Sobolev spaces and their completeness

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Abstract
Sequences spaces $\ell^m_p$, $m \in \mathbb{R}$, $p \in \mathbb{R}_+$ that have called quasi-Sobolev spaces were introduced by Jawad. K. Al-Delfi in 2013 [[1. In this paper, we deal with notion of quasi-inner product space by using concept of quasi-normed space which is generalized to normed space and given a relationship between pre-Hilbert space and a quasi-inner product space with important results and examples. Completeness properties in quasi-inner product space gives us concept of quasi-Hilbert space. We show that, not all quasi-Sobolev spaces $\ell^m_p$, are quasi-Hilbert spaces. The best examples which are quasi-Hilbert spaces and Hilbert spaces are $\ell^m_2$, where $m \in \mathbb{R}$. Finally, propositions, theorems an examples are our own unless otherwise referred.

Keywords: quasi-Sobolev space, quasi-Banach space, Gâteaux derivative, quasi-inner product space, quasi-Hilbert space, smooth quasi-Hilbert space.
1. Introduction

The family of sequence spaces $\ell_p$, $1 < p < \infty$ are normed space where, $\ell_2$ is the only inner product space in this family. Completeness of these spaces can be proved with respect to appropriate norms [2, 3]. Since the triangle inequality fails in the family of sequence spaces $\ell_p$, $0 < p < 1$ where, there is no norm for this range, then imply that it is not Banach space. For a sequence space $\ell_p$, where $0 < p < 1$ and others, many concepts were introduced. One of these concepts is a quasi-Banach space which is based on the definition of a quasi-norm [4]. A quasi-Banach space is a topological linear space [5].

In [1], we were constructed a set of all sequence spaces of power real number $m$, $m \in \mathbb{R}$. The new spaces have called quasi-Sobolev spaces and have denoted by $\ell_p^m$. We were proved that these spaces are quasi-Banach spaces in case $0 < p < \infty$ and they are Banach spaces for $1 < p < \infty$. In our work, we need study these spaces with other concepts such as a pre-Hilbert space and a quasi-inner product space (q.i.p) and their completeness.

In normed spaces, mathematicians have used Gâteaux derivatives to introduce notion of quasi-inner product space and have investigated properties of this concept such as completeness, smoothness and others [6, 7, 8]. This paper is devoted transference above ideology on quasi-normed space to given (q.i.p) and is studied the relationship between this notion and others, in order to study quasi-inner product spaces for $\ell_p^m$ and their completeness.

The paper consists of two sections. Section one includes definitions of quasi-normed space and quasi-Banach space with some useful results which are needed in the section two. One of important theorems which is presented in this section is Jordan-van Neumann theorem. This theorem gives necessary and sufficient conditions to be generated by an inner product space. The second two presents a Gâteaux derivative that has big role to define many concepts, such as quasi-inner product space with completeness property of it. Also, this section shows that this functional is an inner product function in pre-Hilbert spaces. A space $\ell_p^m$, for every $m \in \mathbb{R}$ and $p \in \mathbb{R}_+$ is a quasi-Hilbert space if it is a quasi-inner product space. Hence, with $\ell_p^m$, we find spaces which are quasi-Hilbert spaces and are not Hilbert spaces, spaces neither quasi-Hilbert spaces nor Hilbert spaces and spaces are quasi-Hilbert spaces and Hilbert space.

2. Quasi-normed spaces of sequence spaces.

This section contains notions such as quasi-normed space, a pre-Hilbert space and others with the relationship between them. Also, theorems and equations which are useful in section two are introduced.

Definition 1.1. [4]:
A quasi-norm $\| \cdot \|$ on vector space $V$ over the field of real numbers $\mathbb{R}$ is a function $\| \cdot \| : V \rightarrow [0, +\infty)$ with the properties:

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(1) \( q \| v \| \geq 0, \quad \forall v \in V, \quad q \| v \| = 0 \iff v = 0. \)

(2) \( q \| \alpha v \| = |\alpha| q \| v \| , \quad \forall v \in V, \quad \forall \alpha \in \mathbb{R}. \)

(3) \( q \| v + w \| \leq C \left( q \| v \| + q \| w \| \right) \quad \forall v, w, \in V, \) where \( C \geq 1 \) is a constant independent of \( v, w. \)

A quasi-normed space is denoted by \( (V, q \| \cdot \|) \) or simply \( V. \)

A function \( q \| \cdot \| \) be a norm if \( C = 1, \) thus it is generalization of norm. Every norm function is quasi-norm. The converse does not hold, in general.

Since every quasi-normed space \( V \) is a metric space by \( d(v, w) = q \| v - w \|, \) then it is atopological linear space and the concepts of fundamental sequences and completeness in quasi-normed spaces are given [5]. A quasi-Banach space is a complete quasi-normed space.

**Definition 1.2.**

A symmetric linear functional on \( V^2 \) is a functional \( L \) such that:

(1) \( L(\beta v + \mu w, u) = \beta L(v, u) + \mu L(w, u); \)

(2) \( L(v, w) = L(w, v), \quad \forall \beta, \mu \in \mathbb{R}, \forall v, w, u \in V. \)

**Remark 1.3.**

It is obvious, any inner product function satisfies definition 1.2 and generates a quasi-norm which is \( q \| v \| = (v, v)^{1/2}, \forall v \in V. \)

**Lemma 1.4.**

In a pre-Hilbert space \( V, \) one has the equality:

\[ q \| v + w \|^4 - q \| v - w \|^4 = 8(q \| v \|^2 + q \| w \|^2)^2 \quad \forall v, w, \in V. \]  

**Proof:**

Using remark 1.3, we get \( q \| v + w \|^2 = \langle v + w, v + w \rangle = q \| v \|^2 + 2 \langle v, w \rangle + q \| w \|^2. \)

Also, \( q \| v - w \|^2 = q \| v \|^2 - 2 \langle v, w \rangle + q \| w \|^2 \Rightarrow q \| v - w \|^2 = \left( q \| v \|^2 + q \| w \|^2 \right)^2 - 4(q \| v \|^2 + q \| w \|^2) + 4q \langle v, w \rangle^2. \)

Thus, \( q \| v + w \|^4 - q \| v - w \|^4 = 8(q \| v \|^2 + q \| w \|^2) \quad \forall v, w \in V \) and this is the desired result.

**Definition 1.5.** [1]:

Let \( \{\lambda_k\} \subset \mathbb{R}^+ \) is monotonically increasing sequence such that \( \lim_{K \to \infty} \lambda_k = +\infty, \) quasi-Sobolev spaces are sequence spaces \( \ell^m_p, \) where \( 0 < p < \infty \) and \( m \in \mathbb{R} \) which are defined as

\[ \ell^m_p = \left\{ v = \{v_k\} : \sum_{k=1}^\infty \lambda_k^m |v_k|^p < +\infty \right\}. \]

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When \( m = 0 \) then \( \ell^0_p = \ell_p \), \( 0 < p < \infty \).

**Theorem 1.6.** [1]:
For every \( m \in \mathbb{R} \) and \( p \in \mathbb{R}^+ \), a space \( \ell^m_p \), is a quasi-Banach space with the function:
\[
q\|v\| = \left( \sum_{k=1}^\infty \lambda_k^{mp} |v_k|^p \right)^{1/p}.
\]

We note that the constant \( C = 2^{1/p} \) for \( p \in (0, 1) \), and \( C = 1 \) for \( p \in [1, +\infty) \).

**Theorem 1.7.** (parallelogram equality)
Let \( V \) be a pre-Hilbert space. Then \( \forall v, w \in V \),
\[
q\|v + w\|^2 + q\|v - w\|^2 = 2q\|v\|^2 + 2q\|w\|^2
\]
(2)  

**Proof:**

Since \( V \) be a pre-Hilbert space and \( <v, w> = \left( \frac{1}{4} q\|v + w\|^2 - \frac{1}{4} q\|v - w\|^2 \right) \) from remark 1.3 and proof of lemma 1.4, then putting this function in equation (1) we obtain the desired result.

Now, we introduce Jordan-van Neumann theorem in quasi-normed spaces.

**Theorem 1.8.** (Jordan – van Neumann)
A quasi-normed space \( V \) is a pre-Hilbert space iff equality (2) is satisfied by the quasi-norm of \( V \).

**Proof:**
The proof of this theorem is very technical and proceeds in a way similar to its version in normed space (see [3]).

The next example shows the importance of the parallelogram equality mentioned in the previous theorem.

**Example 1.9:**
Let \( v \) and \( w \) belong to the quasi-normed space \( \ell_{1/2}^{-1} \), where \( v = \{v_k\} = \{0.1, 0, 0, 0, \ldots\} \), \( w = \{w_k\} = \{0, 0.2, 0, 0, \ldots\} \) and take \( \{\lambda_k\} = \{k\} \), \( k \in \mathbb{N} \). Then we have:
\[
\frac{1}{\sqrt{2}} \|v + w\|^2 = \left( \sum_{k=1}^\infty \lambda_k^{-\frac{1}{2}} |x_k + y_k|^{1/2} \right) = 0.479262779275938 = \frac{1}{\sqrt{2}} \|v - w\|^2,
\]
so
\[
\frac{1}{\sqrt{2}} \|v + w\|^2 + \frac{1}{\sqrt{2}} \|v - w\|^2 = 0.958525584551875, \quad \text{and,} \quad 2 \frac{1}{\sqrt{2}} \|v\|^2 + 2 \frac{1}{\sqrt{2}} \|w\|^2 = 0.482842712474619.
\]
It is clear that two sides of the equation (2) do not hold. Thus, \( \ell_{1/2}^{-1} \) is not pre-Hilbert space.

3. Quasi-inner product spaces of sequence spaces

A Gâteaux derivative is used to define many concepts, such as quasi-inner product function, and smooth quasi-Hilbert space with some important results and examples.

**Definition 2.1.**
Let $V$ be a vector space over the field $\mathbb{R}$ equipped with $q\|\cdot\|$. A Gâteaux derivative of $q\|v\|$ is a functional $\delta(v,w)$ at $v \in V$ in the direction $w \in V$ which is defined as:

$$
\delta(v,w) = (\delta_1(v,w) + \delta_2(v,w))
$$

such that:

$$
\delta_1(v,w) = \lim_{h \to 0^+} h^{-1} (q\|v + hw\| - q\|v\|), \quad \delta_2(x,y) = \lim_{h \to 0^-} h^{-1} (q\|v + hw\| - q\|v\|),
$$

where $h \in \mathbb{R} \setminus \{0\}$. In similar way, we define $\delta(w,v)$.

Gâteaux derivatives $\delta(v,w)$ and $\delta(w,v)$ inspires the functionals $\tau(v,w) = \frac{q\|v\|}{2} \delta(v,w)$ and $\tau(w,v) = \frac{q\|w\|}{2} \delta(w,v)$ sequentially.

**Definition 2.2**

A Gâteaux derivative $\tau(v,w)$ is said to be quasi-inner product function if $\tau(w,v)$ exists and the next equality is satisfied:

$$
\tau(v+w,k)^4 - q\|v-w\|^4 = 8 (q\|v\|^2 \tau(v,w) + q\|w\|^2 \tau(w,v)), \quad \forall \ v, w \in V
$$

(3)

Similarly, $\tau(w,v)$. A space $V$ is said to be a quasi-inner product if both $\tau(v,w)$ and $\tau(w,v)$ are quasi-inner product functions.

**Lemma 2.3**

For every positive integer $p \geq 1$ and $m \in \mathbb{R}$, the functional $\tau(v,w)$ in quasi-Sobolev spaces $\ell_p^m$ exists and is defined as:

$$
\tau(v,w) = q\|v\|^{4-p} \sum_{k} \lambda_k^{\frac{mp}{2}} |v_k|^{p-1}(\text{sng} \ v_k) w_k, \quad \forall \ v \in \ell_p^m \text{ s.t. } q\|v\| \in E,
$$

where, $E = \left\{ q\|v\|: \begin{cases} q\|v\| \geq 0, & P = 1 \\ q\|v\| > 0, & P \geq 2 \end{cases} \right\}$ and

$$
\text{sng} \ v_k = \begin{cases} 1, & v_k > 0 \\ 0, & v_k = 0 \\ -1, & v_k < 0 \end{cases}.
$$

(4)

Similarly, we define $\tau(w,v)$.

**Proof:**

In definition 2.1, we use properties of limits of functions and applying definition of a quasi-norm function of $\ell_p^m$ which is in theorem 1.6 with help of the binomial theorem, which is for every positive integer $p$, $(v+w)^p = \sum_{k=0}^{p} \binom{p}{k} v^k w^{p-k}$, we get Eq. (4).

**Proposition 2.4.**

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The existence of the limit in definition of Gâteaux functions is necessary condition, not sufficient, in order that any quasi-normed space be a quasi-inner product space.

**Proof**
Suppose $V$ is a quasi-normed space. From definition 2.1, we observe that existence of $\delta_1(v,w)$ and $\delta_2(v,w)$ are connected by the limit on behavior of the quasi-norm as $h \to \pm 0$. Hence, $\tau(v,w)$ is exist if this limit is exist. Also, with $\tau(w,v)$ similarly.

To explains above condition is not sufficiently, we take the example:

**Example 2.5:**
Suppose $v, w \in \ell^1_3$, where $v = \{v_k\} = \{1, 0, 0, 0, \ldots\}$, $w = \{w_k\} = \{1, 1, 0, 0, \ldots\}$ and take $\{\lambda_k\} = \{\sqrt{k}\}$, $k \in \mathbb{N}$. Then, using lemma 2.3, we get $\tau(v,w) = 1$, $\tau(w,v) = 0.372884880824589$. Thus, $\tau(v,w)$ and $\tau(w,v)$ are exist. However, equation (3) is not satisfied. Therefore, the space $\ell^1_3$ is not quasi-inner product space.

**Remark 2.6.**
If cases the values of $p$ differ from those values considered in lemma 2.3, we have quasi-Sobolev spaces $\ell^m_p$ which are not quasi-inner product. For instance, in case $p \in (0,1)$, as it is shown in the example 1.9. Indeed, with the space $\ell^{-1}_{1/2}$, $\delta_1(v,w)$ and $\delta_2(w,v)$ do not exist, since there is no limit as $h \to \pm 0$ from definition 2.1. Then right hand in Eq. (3) is not finite, while left hand equal zero.

**Definition 2.7**
A quasi-normed space $V$ is smooth if $\delta_1(v,w)$ and $\delta_2(v,w)$ have one value.

When $V$ is smooth quasi-normed space, then $\tau(v,w) = \langle v, w \rangle$ and $\tau(w,v) = \langle w, v \rangle$. Hence, we have equation (3), and the definition 2.2 is hold. Thus, $V$ is an quasi-inner product space.

**Proposition 2.8.**
Every pre-Hilbert space is a quasi-inner product space.

**Proof:**
Let $V$ is a pre-Hilbert space. According to lemma 1.4, an inner product function gives eq. (1). Also, By remark 1.3 and definition 2.1, we obtain $\tau(v,w) = \langle v, w \rangle$ and $\tau(w,v) = \langle w, v \rangle$. Hence, we have equation (3), and the definition 2.2 is hold. Thus, $V$ is an quasi-inner product space.

The converse of proposition does not hold, consider the following example:

**Example 2.9:**
Take example 2.5 with replace space $\ell^1_3$ by $\ell^1_4$. Since Eq. (3) is satisfied with quasi-normed space $\ell^1_4$, where the left and right hand of Eq. (3) are equal to 16, so it is quasi-inner product space. But the left and right hand of Eq. (2) are not equal, hence this space is not a pre-Hilbert space.

**Definition 2.10.**
A complete quasi-inner product space is called a quasi-Hilbert space.

If a quasi-Hilbert space is smooth, then it is called a smooth quasi-Hilbert space.

We recall that completeness property is coming from this property of quasi-normed space.

**Theorem 2.11.**
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For every $m \in \mathbb{R}$, $\ell_2^m$ is a smooth quasi-Hilbert space and Hilbert space.

Proof:
According to lemma 2.3, we get $\tau(v,w) = \sum \lambda_k^m |v_k|(\text{sng } v_k) w_k$, and $\tau(w,v) = \sum \lambda_k^m |w_k|(\text{sng } w_k) v_k$ which are linear by definition 1.2, with definition of $\tau(v,w)$ and $\tau(w,v)$ as above, then they are symmetric, that is, $\tau(v,w) = \tau(w,v)$, and $\tau(v,v) = \|v\|^2 \geq 0$, with equality iff $v = 0$. Hence, $\ell_2^m$ is a pre-Hilbert space. By proposition 2.8, it is a quasi-inner product space, where $8 \sum \lambda_k^m |v_k|^3(\text{sng } v_k) w_k + 8 \sum \lambda_k^m |w_k|^3(\text{sng } w_k) v_k$ is value to both sides of equation (3). If we apply quasi-norm function of $\ell_2^m$ in definition 2.1, we obtain $\delta_1(v,w) = \delta_2(v,w)$ since the limit in $\delta_1(v,w)$ itself one $\delta_2(v,w)$. Then $\ell_2^m$ is smooth.

Now, since $\ell_2^m$ is a quasi-Banach space for every $m \in \mathbb{R}$ by theorem 1.6, then it is complete under $\|v\| = (\tau(v,v))^{1/2}$, i.e. every fundamental sequence $\{v_k\}$, $k \in \mathbb{N}$ is convergent in it. Therefore, Theorem is proved.

Remark 2.12.
Since a space $\ell_p^m$, for every $m \in \mathbb{R}$ and $p \in \mathbb{R}_+$ is a quasi-Banach space, then $\ell_p^m$ is a quasi-Hilbert space if it is a quasi-inner product space.

References